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The dynamics of vortex structures and states of current in plasma-like fluids and the electrical explosion of conductors: I. The model of a non-equilibrium phase transition

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Abstract. A set of equations according to which the conducting medium consists of two fluids—laminar and vortex—has been obtained by transforming the MHD equations. In a similar way, an electronic fluid is assumed to consist of a laminar and a vortex fluid. This system allows one to study the formation and dynamics of large-scale hydrodynamic fluctuations. From this model, a model of a non-equilibrium phase transition belonging to a class of the Lorenz-type models has been developed. Vortex structures resulting in the increase in an effective resistance of the conducting medium and the interruption of current have been shown to appear even at constant transport coefficients in a laminar electronic fluid. Critical exponents of the order parameters (amplitudes), which for a direct current coincide with the critical exponents in the Lorenz model, have been found. A spatial scale of the structure described by the theory is in good agreement with experiment. A further evolution of vortex structures has been shown to occur by splitting the spatial scale. A similarity, according to which the following sequence of splitting takes place: $k_0 \to k_1 = 0.5k_0 \to k_2 = 2k_0 \to k_3 = 2k_2$, etc, has been hypothesized.

1. Introduction

The present series of three papers describes the dynamics of vortex structures and states of current in plasma-like fluids as a magnetohydrodynamic (MHD) approximation. Vortex structures are large-scale (hydrodynamic) fluctuations, obeying the condition $kL \geqslant 1$, where k and L are, respectively, the wavenumber and the characteristic size (for instance, the sample size). Criteria for the classification of fluctuations into small-scale (kinetic) with $kL \ll 1$ and large-scale are discussed in [1]. In addition, a local thermodynamic equilibrium (LTE) is valid, and local kinetic transport coefficients (σ is the electric conductivity, κ is the thermal conductivity, η is the shear viscosity) are still determined by kinetic fluctuations.

Below, in section 3, and also in the second paper of the series, we show that even in the case of a constant transport coefficient, the emergence of vortex hydrodynamic structures results in a spontaneous breaking of a symmetry of a laminar, in the initial state, 'electron' fluid and the appearance of vortex current structures. As a consequence, the current in a cylindrical conductor is interrupted, i.e. an effective resistance of the conductor $R_{\text{eff}} \to \infty$ and a region of negative differential resistance appears in the UI characteristic. This process is singular in time and for its realization it is necessary that the external electric circuit should be supercritical to some extent, which is a characteristic property of a non-equilibrium phase transition (NPT) [2].

Section 3 presents a simple NPT model for a cylindrical conductor. To develop it we used the set of magnetohydrodynamic equations (MHD) obtained in section 2. We also showed that the formation of vortex structures in the medium, from the view point of an external observer results in an effective alteration in kinetic coefficients, and local kinetic coefficients determined by small-scale fluctuation as before.

In the second paper of this series we used computer simulation to study the dynamics of vortex structures and current states in electrophysical systems where the NPT model from section 3 is used as a model of a nonlinear element (NE).

In the third paper we discuss and give a qualitative explanation of the electrical explosion in conductors (EEC) [3, 4], which is an example of a phenomenon where vortex structures play, in our opinion, the dominant role (see [5], which draws attention to the analogy between initial stages of EEC and the turbulence in an incompressible liquid). During an electrical explosion the conductor first breaks down into transverse strata (axial structurization) and then it expands forming a low-temperature plasma with a condensed disperse phase (CDP) by the end of the explosion. It has been established experimentally [4] that the conductor stratification takes a time that is less than a characteristic hydrodynamic time; in particular, it is less than the sound time, $t_s = r_0 c_s{}^{-1}$ (r_0, c_s are the conductor radius and the sound velocity, respectively). The above fact, and other experimental evidence previously considered anomalous, find a natural explanation in the framework of our model.

2. Mathematical model

For the initial mathematical model we use hydrodynamic equations of a conducting liquid and Maxwell equations [6], restricting ourselves to the MHD approximation. We assume that there is LTE in the system. As noted in [7], physical processes resulting in the infringement of LTE make a major contribution to the bulk viscosity; therefore, proceeding from the assumption of LTE, the bulk viscosity is not considered in the set of equations below (the existence of large-scale vortex excitations in the medium leads to physical effects that can be explained as the influence of an effective bulk viscosity).

In this case the set of MHD equations becomes

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{u}) = 0 \tag{1}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} - [\boldsymbol{u}, [\nabla, \boldsymbol{u}]] + \nabla \frac{u^2}{2}\right) = -\nabla P + \frac{1}{4\pi}[[\nabla, \boldsymbol{H}], \boldsymbol{H}] + \nabla \cdot \hat{\mathbf{S}} \tag{2}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \nabla)T\right)$$
$$= -P_T(\nabla \cdot \boldsymbol{u}) + \frac{\nu_m}{4\pi}([\nabla, \boldsymbol{H}])^2 + +Sp(\hat{\mathbf{S}} \cdot (\nabla \otimes \boldsymbol{u})^{\mathrm{T}}) + (\nabla \cdot \kappa \nabla T) \tag{3}$$

$$[\nabla, \boldsymbol{E}] = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} \tag{4}$$

$$[\nabla, \boldsymbol{H}] = \frac{4\pi}{c}\boldsymbol{j} \tag{5}$$

$$(\nabla \cdot \boldsymbol{H}) = 0 \tag{6}$$

$$(\nabla \cdot \boldsymbol{E}) = 4\pi\delta\rho_e \tag{7}$$

where ρ, $\boldsymbol{u}$, T, P, $\boldsymbol{E}$, $\boldsymbol{H}$, $\delta\rho_e$ and $\boldsymbol{j}$ are, respectively the local density, the hydrodynamic velocity, the temperature, the pressure, the electric intensity, the magnetic intensity, the density of a non-balanced electric charge (in the MHD approximation $\langle\rho_e\rangle = 0$; equation (7) is used below to find the deviation $\delta\rho_e$ from the mean charge value induced by an effective polarization of the conducting medium) and the current density. To find the latter, we use the generalized Ohm's law in the simplest form

$$\boldsymbol{j} = \sigma\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}, \boldsymbol{H}]\right) \tag{8}$$

$P_T = T(\partial P/\partial\rho)_T$; $\hat{\mathbf{S}} = 2\eta(\hat{\mathbf{U}} - 3^{-1}(\nabla \cdot \boldsymbol{u})\hat{\mathbf{I}})$ is the tensor of a viscous stress, $\hat{\mathbf{U}} = 0.5((\nabla \otimes \boldsymbol{u}) + (\nabla \otimes \boldsymbol{u})^{\mathrm{T}})$, $\hat{\mathbf{I}}$ is the unit tensor (the symbol T denotes a transposition; $\cdot$ is the internal (scalar) vector product, $\otimes$ is the tensor product of vectors; $[\boldsymbol{a}, \boldsymbol{b}]$ is the vector product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; $\nabla = \nabla_i e^i$ is the gradient operator); c_ρ is the heat capacity per unit mass; $\nu_m = c^2(4\pi\sigma)^{-1}$ is the magnetic viscosity, c is the velocity of light.

To close the set of equations (1)–(8), it is necessary to specify initial and boundary conditions. In our case, it is sufficient to set the value of a normal

$$P - \hat{S}_{nn} = 0 \tag{9}$$

and a tangential

$$\hat{S}_{n\tau} = 0 \tag{10}$$

component of the stress tensor and the value of the magnetic intensity on a free boundary between the medium and vacuum ($\hat{S}_n = \hat{\mathbf{S}}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector to the conductor surface).

We rewrite (1)–(8) in the form where large-scale vortex excitations can be explicitly represented. It is remarkable that a significant role in their formation is played by a surface that restricts the volume occupied by a continuous conducting medium. This surface enables distinguishing vector fields (of hydrodynamic velocity and current density) which are connected to themselves inside and outside this volume, or end on the boundary (conventionally called 'laminar' and 'vortex' vector fields).

Actually, let

$$\boldsymbol{u} = \boldsymbol{u}_\rho + \boldsymbol{u}_\omega \tag{11}$$

with

$$(\nabla \cdot \boldsymbol{u}_\omega) = 0 \qquad [\nabla, \boldsymbol{u}_\rho] = 0 \tag{12}$$

where $\boldsymbol{u}_\rho$, $\boldsymbol{u}_\omega$ are the laminar and the vortex vector fields of hydrodynamic velocity, respectively. We write the motion equation of a vortex liquid in the following form

$$\rho\left(\frac{\partial \boldsymbol{u}_\omega}{\partial t} - [\boldsymbol{u}_\omega, [\nabla, \boldsymbol{u}_\omega]] + \nabla\frac{{u_\omega}^2}{2}\right) = -\nabla(P_\omega - P_m) + \frac{1}{4\pi}[[\nabla, \boldsymbol{H}], \boldsymbol{H}] + \nabla \cdot \hat{\mathbf{S}}_\omega \tag{13}$$

where $P_m = H^2(8\pi)^{-1}$ is the magnetic pressure; $\hat{\mathbf{S}}_\omega = 2\eta\hat{\mathbf{U}}_\omega$; $\hat{\mathbf{U}}_\omega = 0.5(\nabla \otimes \boldsymbol{u}_\omega + (\nabla \otimes \boldsymbol{u}_\omega)^{\mathrm{T}})$; P_ω is the hydrodynamic pressure, the spatial distribution of which is derived from the solution of the Poisson-type equation found by using the first relation of (12) in (13)

$$\begin{aligned}\Delta(P_\omega - P_m) &- \nabla(\ln\rho) \cdot \nabla(P_\omega - P_m) \\ &= \rho\nabla \cdot \left(-\nabla\frac{{u_\omega}^2}{2} + [\boldsymbol{u}_\omega, [\nabla, \boldsymbol{u}_\omega]] + \rho^{-1}\left((4\pi)^{-1}[\nabla, [\nabla, \boldsymbol{H}]] + \nabla \cdot \hat{\mathbf{S}}_\omega\right)\right).\end{aligned} \tag{14}$$

By substituting (12) into (1), (2) with the consideration of (13), we obtain

$$\frac{\partial\rho}{\partial t} + (\boldsymbol{u}_\rho \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{u}_\rho) = -(\boldsymbol{u}_\omega \cdot \nabla)\rho \tag{15}$$

$$\rho\left(\frac{\partial \boldsymbol{u}_\rho}{\partial t} + \nabla\frac{{u_\rho}^2}{2}\right) = -\nabla(P_\rho + P_\mathrm{m}) - \rho(\nabla(\boldsymbol{u}_\rho \cdot \boldsymbol{u}_\omega) - [\boldsymbol{u}_\rho, [\nabla, \boldsymbol{u}_\omega]]) + \nabla \cdot \hat{\mathbf{S}}_\rho \tag{16}$$

where $P_\rho = P - P_\omega$, $\hat{\mathbf{S}}_\rho = 2\eta(\hat{\mathbf{U}}_\rho - 3^{-1}(\nabla \cdot \boldsymbol{u}_\rho)\hat{\mathbf{I}})$, $\hat{\mathbf{U}}_\rho = 0.5(\nabla \otimes \boldsymbol{u}_\rho + (\nabla \otimes \boldsymbol{u}_\rho)^\mathrm{T})$. From the viewpoint of an external observer the second term in the right-hand side of (16) has the physical sense of an effective volume viscosity.

As is consistent with the second relation from (12), equation (16) has the form

$$\frac{\partial \boldsymbol{u}_\rho}{\partial t} = -\nabla\left(\frac{u^2}{2} + \chi\right) \tag{17}$$

where χ is the effective chemical potential, the spatial distribution of which is derived from the Poisson-type equation found by using expressions (12) and (14) in equation (16)

$$\begin{aligned}\Delta\chi = \nabla \cdot \Big(\rho^{-1}\big(\nabla(P - \tfrac{4}{3}\eta(\nabla \cdot \boldsymbol{u}_\rho)) - (4\pi)^{-1}[[\nabla, \boldsymbol{H}], \boldsymbol{H}] \\ + [\nabla, \eta[\nabla, \boldsymbol{u}_\omega]] + 2(\nabla \cdot \boldsymbol{u}_\rho)\nabla\eta - 2\nabla\eta \cdot (\nabla \otimes \boldsymbol{u})^\mathrm{T}\big) - [\boldsymbol{u}, [\nabla, \boldsymbol{u}_\omega]]\Big).\end{aligned} \tag{18}$$

In the region of developed large-scale fluctuations the constraint equation (18) should be used as an effective equation of state of liquid. For a homogeneous incompressible liquid with $\eta = 0$, $|\boldsymbol{H}| = 0$ and $|\boldsymbol{u}_\omega| = 0$, the solution (18) has the form $\chi = P\rho^{-1}$.

The boundary conditions are rewritten as

$$P_\rho - \hat{S}_{\rho,nn} = -(P_\omega - \hat{S}_{\omega,nn}) \tag{19}$$

$$\hat{S}_{\rho,n\tau} = -\hat{S}_{\omega,n\tau}. \tag{20}$$

Hence, the existence of large-scale vortex excitations in the liquid results in the appearance of back pressure and tangent stress on the free boundary with a vacuum, which prevents a change in the shape of the volume occupied by the liquid.

Vortex current excitations can also appear in plasma-like media, therefore we transform Maxwell's equations (4)–(7), taking Ohm's law into account, in the form of (8). Let $\boldsymbol{j} = \boldsymbol{j}_\rho + \boldsymbol{j}_\omega$, with

$$\int_\Sigma \boldsymbol{j}_\rho \cdot \mathrm{d}\boldsymbol{s} = I \qquad \int_\Sigma \boldsymbol{j}_\omega \cdot \mathrm{d}\boldsymbol{s} = 0 \tag{21}$$

(I is the total current, Σ is the surface restricting the volume occupied by the conducting medium). Then

$$\boldsymbol{j}_\rho = \sigma(\boldsymbol{E}_\rho + c^{-1}[\boldsymbol{u}, \boldsymbol{H}_\rho]) \tag{22}$$

$$\boldsymbol{j}_\omega = \sigma(\boldsymbol{E}_\omega + c^{-1}[\boldsymbol{u}, \boldsymbol{H}_\omega]) \tag{23}$$

$$[\nabla, E_\rho] = -c^{-1}\frac{\partial H_\rho}{\partial t} \tag{24}$$

$$[\nabla, E_\omega] = -c^{-1}\frac{\partial H_\omega}{\partial t} \tag{25}$$

$$[\nabla, H_\rho] = 4\pi c^{-1} j_\rho \tag{26}$$

$$[\nabla, H_\omega] = 4\pi c^{-1} j_\omega \tag{27}$$

$$(\nabla \cdot E_\rho) + (\nabla \cdot E_\omega) = 4\pi\delta\rho_e \tag{28}$$

$$(\nabla \cdot H_\rho) + (\nabla \cdot H_\omega) = 0. \tag{29}$$

In the absence of vortex current excitations, i.e. at $|j_\omega| = 0$, $(\nabla \cdot H_\rho) = 0$. Therefore, to exclude non-physical solutions, we require that

$$(\nabla \cdot H_\rho) = 0 \tag{30}$$

$$(\nabla \cdot H_\omega) = 0. \tag{31}$$

Then a set of diffusion equations H_ρ and H_ω follows from (21)–(27) and (30), (31)

$$\frac{\partial H_\rho}{\partial t} + (u_\rho \cdot \nabla)H_\rho = (H_\rho \cdot \nabla)u_\rho - H_\rho(\nabla \cdot u_\rho) + \{(H_\rho \cdot \nabla)u_\omega - (u_\omega \cdot \nabla)H_\rho\} + \nu_m \Delta H_\rho - [\nabla \nu_m, [\nabla, H_\rho]] \tag{32}$$

$$\frac{\partial H_\omega}{\partial t} + (u_\omega \cdot \nabla)H_\omega = (H_\omega \cdot \nabla)u_\omega - H_\omega(\nabla \cdot u_\rho) + \{(H_\omega \cdot \nabla)u_\rho - (u_\rho \cdot \nabla)H_\omega\} + \nu_m \Delta H_\omega - [\nabla \nu_m, [\nabla, H_\omega]]. \tag{33}$$

The terms between the brackets in (32) and (33) consider the influence of laminar and vortex vector fields of the hydrodynamic velocity on the alteration in magnetic fields. This influence can be recognized by an external observer as an effective variation in the magnetic diffusion factor. It should also be noted that a direct measurement of the field H_ω is impossible due to condition (21). However, it contributes to an effective voltage drop across a conductor, thus determining the total current I (see section 3).

The energy balance equation (3) remains unaltered, the work of pressure forces to expand (compress) the matter being determined by the field u_ρ, and local kinetic transport factors by small-scale fluctuations.

Let us make some comments. The equations obtained are a phenomenological set of equations for a two-liquid hydrodynamic description of a 'heavy' conducting fluid and a 'light' fluid of magnetic field. In this framework a simple model of a non-equilibrium phase transition (section 3) has been developed and the dynamics of vortex structures and states of current in electrophysical circuits has been studied. While deriving (13)–(18) and (32), (33) we did not use additional physical considerations which were beyond the applicability of the initial model (1)–(8). In addition, an essential condition for the consistency of motions determined by this system is to satisfy the condition of constraint (18), which is similar to the condition of deformation compatibility in elastic theory. From the view point of an external observer, characteristics of the field u_ω can be regarded as additional thermodynamic variables which enter a thermodynamic potential as independent variables. To describe the dynamics of the latter, it is necessary to derive relaxation-type equations [7, 8] and to use experimental data for finding a relaxation time. In the framework of the field approach developed in this paper, the above problem no longer arises since relaxation times will appear only when equations (13), (14) are reduced to relaxation-type equations.

3. The model of a non-equilibrium phase transition

A homogeneous incompressible liquid is used in the development of the NPT model. This fluid occupies a region of fixed size (as is shown in [5], it is correct for the initial stage of EEC; discussed below is a liquid conducting cylinder with radius r_0 and length $l \gg r_0$). In this case $|\boldsymbol{u}_\rho| = 0$ and the compatibility condition (17) is satisfied identically. Moreover, as in [5], we assume local kinetic coefficients to be constant and consequently we do not take into consideration, in a first approximation, the influence of the conductor heating on the dynamics of vortex structures and states of current.

The set of equations (11)–(16) and (32), (33) is then written in the following form (below we do not use the index ω since we discuss a vortex fluid; moreover, instead of (32) and (33), we use a diffusion equation for the total magnetic field $\boldsymbol{H}$)

$$(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = 0 \tag{34}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -{\rho_0}^{-1}\boldsymbol{\nabla} P + (4\pi\rho_0)^{-1}[[\boldsymbol{\nabla}, \boldsymbol{H}], \boldsymbol{H}] + \nu\Delta\boldsymbol{u} \tag{35}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{H} = (\boldsymbol{H} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu_{\mathrm{m}}\Delta\boldsymbol{H} \tag{36}$$

where $\nu = \eta{\rho_0}^{-1}$ is the kinematic viscosity and Δ is the Laplacian. As follows from (34), $\boldsymbol{u} = [\boldsymbol{\nabla}, \boldsymbol{A}]$ ($\boldsymbol{A}$ is the vector potential of the velocity, which is known to be accurate within the gradient of a scalar function; below we use the Coulomb gauge of the vector potential $(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = 0$).

We direct the z axis along the axis of the conductor and we make use of the azimuthal symmetry, setting $\boldsymbol{u} = \{u_r(r, z, t), 0, u_z(r, z, t)\}$, $\boldsymbol{H} = \{0, H(r, z, t), 0\}$, $\boldsymbol{A} = \{0, \psi(r, z, t), 0\}$. From (34) we then find $u_r = -\partial\psi/\partial z$, $u_z = \partial(r\psi)/r\partial r$. We look for a solution of (36) in the form $H(r, z, t) = H_1(r, t) + h(r, z, t)$, where $H_1(r, t) = 2i(t)r(c{r_0}^2)^{-1}$, i.e. while finding H_1, the current density is assumed to be homogeneous in the cross section of the conductor. In this case the boundary conditions for (36) are satisfied automatically, since $H_1(r_0, t) = 2i(t)(cr_0)^{-1}$. Correspondingly, the boundary conditions for the field $h(r, z, t)$ at $r = 0$ and $r = r_0$ are zero: $h(0, z, t) = h(r_0, z, t) = 0$. Applying the vector operation $[\boldsymbol{\nabla}, \ ]$ to (35) and making use of the above assumptions, we derive a set of differential equations in partial derivatives for ψ and h

$$\begin{aligned}\frac{\partial(\Delta\psi - \psi r^{-2})}{\partial t} &= -\frac{\partial(\psi, \Delta\psi - \psi r^{-2})}{\partial(r, z)} + \frac{2(\Delta\psi - \psi r^{-2})}{r}\frac{\partial\psi}{\partial z} \\ &\quad + R\frac{\nu\nu_{\mathrm{m}}}{H_0{r_0}^3}\frac{i}{I_0}\frac{\partial h}{\partial z} + \nu[\Delta(\Delta\psi - \psi r^{-2}) - r^{-2}(\Delta\psi - \psi r^{-2})]\end{aligned} \tag{37}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial(\psi, h)}{\partial(r, z)} + \frac{H_0}{r_0}\frac{i}{I_0}\frac{\partial\psi}{\partial z} + \nu_{\mathrm{m}}\left(\Delta h - \frac{h}{r^2}\right) \tag{38}$$

where

$$\frac{\partial(a, b)}{\partial(r, z)} = \frac{\partial(ra)}{r\partial r}\frac{\partial b}{\partial z} - \frac{\partial a}{\partial z}\frac{\partial(rb)}{r\partial r}$$

$R = {H_0}^2{r_0}^2(2\pi\rho_0\nu_{\mathrm{m}}\nu)^{-1} = {v_{\mathrm{A}}}^2{r_0}^2(\nu_{\mathrm{m}}\nu)^{-1} = {Pe_{\mathrm{m}}}^2 s^{-1}$ is the Rayleigh number, $v_{\mathrm{A}} = H_0(2\pi\rho_0)^{-1/2}$ is the Alfven velocity, $Pe_{\mathrm{m}} = v_{\mathrm{A}}r_0{\nu_{\mathrm{m}}}^{-1}$ is the magnetic Peclet number,

$s = \nu \nu_m^{-1}$, $H_0 = 2I_0(cr_0)^{-1}$, I_0 is the characteristic current value dependent on the type of energy source. For the direct current, $i(t) = I_0 =$ constant, the set of equations (37), (38), as shown in [5], agrees accurately with the replacement of a Cartesian coordinate system by a cylindrical one and the replacement of the heat conduction equation for the diffusion of a magnetic field, with the set of Saltzman equations [9] in the theory of the Benard effect [10].

A second term in the right-hand side of (37) leads to the breakdown of azimuthal symmetry (leading to dependence of the fields ψ and h on φ). Below we limit ourselves to the consideration of instabilities which do not lead to the breakdown of the azimuthal symmetry. For this reason we ignore this term.

Equations (37), (38) without a second term in the right-hand side of (37) describe the dynamics of large-scale fluctuations in an incompressible conducting medium. If we make a linear analysis of the stability of their solution, we can reveal that it is unstable and we can anticipate that the development of the most rapid perturbations is determined by a minimal number of modes, in particular by three.

Let us show that the set of equations (37), (38) can be reduced to the set of three nonlinear differential equations that describe the interaction of the three perturbation modes (later on we will discuss the plausibility of this suggestion). Similarly to [5, 10], we restrict ourselves to one perturbation mode for ψ and to two modes for h and also, following Lorenz [10], we assume free boundary conditions for ψ: $\psi(0, z, t) = \psi(r_0, z, t) = 0$. Retaining the less significant terms in the Fourier representation of ψ and h, we use the substitution, which transforms into the Lorenz substitution in case of a Cartesian coordinate system

$$\psi(r, z, t) = \sqrt{2}\frac{1 + (\pi k/g_1)^2}{k}\nu_m X(t)\sin\left(\frac{\pi k z}{r_0}\right)J_1\left(\frac{g_1 r}{r_0}\right) \tag{39}$$

$$h(r, z, t) = \frac{H_0}{\pi r_1}\left(\sqrt{2}Y(t)\cos\left(\frac{\pi k z}{r_0}\right)J_1\left(\frac{g_1 r}{r_0}\right) - 2Z(t)J_0\left(\frac{g_1 r}{r_0}\right)J_1\left(\frac{g_1 r}{r_0}\right)\right). \tag{40}$$

In (39), (40) $g_1 = 3.83171$ corresponds to the first zero of the Bessel function $J_1(x)$, $R_c = 64 g_1^2 \pi^2 (b(4 - b))^{-1}$ is the critical Rayleigh number, $b = 4(1 + (\pi k g_1^{-1})^2)^{-1}$. By substituting (39), (40) for (37), (38), we find a set of ordinary differential equations for the $X(t), Y(t)$ and $Z(t)$ amplitudes, which coincides with the set of equations (4) in [5] at $i(t) = I_0 =$ constant (note that in [5] $ZJ_1(2g_1 r r_0^{-1})$ is used for h instead of $2ZJ_0(g_1 r r_0^{-1})J_1(g_1 r r_0^{-1})$; this, however, does not transform the system of amplitudes obtained in [5], as substitution (21*b*), corresponding to the approximate equality $J_1(2x) \cong 2J_0(x)J_1(x)$, was actually used in their derivation)

$$\dot{X} = s(-X + IY) \tag{41}$$

$$\dot{Y} = \pi g_1^{-1} X(-Z + \pi g_1^{-1} r_1 I) - Y \tag{42}$$

$$\dot{Z} = -(\pi g_1^{-1} XY + bZ) \tag{43}$$

where the point symbol '$\cdot$' is used to denote the differentiation operator for the dimensionless time $\tau = 4 g_1^2 b^{-1} \nu_m t r_0^{-2}$, $I(t) = i I_0^{-1}$ is the dimensionless current; $r_1 = R R_c^{-1}$ is the control parameter of the model.

The knowledge of $X(t), Y(t), Z(t)$ and $I(t)$ allows us to determine paths of particles transferring the mass (of 'hydrodynamic' particles, below called atoms) and the current (of 'conducting electrons'), while solving the motion equation

$$\dot{R}_a = -C_r X\cos(\pi k Z_a)J_1(g_1 R_a) \tag{44}$$

$$\dot{Z}_a = C_z X\sin(\pi k Z_a)J_0(g_1 R_a) \tag{45}$$

and

$$\dot{R}_e = \sqrt{2}Bkr_1^{-1}Y\sin(\pi kZ_e)J_1(g_1R_e) \tag{46}$$

$$\dot{Z}_e = B\left[2I + ar_1^{-1}\left(\sqrt{2}Y\cos(\pi kZ_e)J_1(g_1R_e) + 2Z(J_1^2(g_1R_e) - J_0^2(g_1R_e))\right)\right] \tag{47}$$

where R_a, Z_a and R_e, Z_e are the coordinates of paths for atoms and conducting electrons, respectively; $C_r = 2^{1/2}\pi g_1^{-2}$; $C_z = g_1k^{-1}C_r$; $B = bcH_0(16\pi en_e\nu_m g_1^2)^{-1}$; $a = g_1\pi^{-1}$; e and n_e are, respectively, the unit charge and the density of conducting electrons. Since (44)–(47) determine paths of particles corresponding to every moment of time τ, then in their integration the X, Y, Z and I amplitudes should be assumed to be constant, i.e. systems (44), (45) and (46), (47) represent, on the plane (r, z), autonomous dynamic systems of second order. Moreover, the set of equations (41)–(43), combined with additional equations for the current I, is a control dynamic system for sets (44), (45) and (46), (47). Thus we distinguish between 'slow' processes, the dynamics of which is determined by system (41)–(43), and 'rapid' ones, the dynamics of which is determined by systems (44), (45) and (46), (47). Therefore, at certain values of X, Y, Z and I in dynamic systems (44), (45) and (46), (47), one should anticipate bifurcations which can result in a topological rearrangement of spatial structures, in particular in a spontaneous breaking of symmetry.

To find the value of the current in the conductor, we should specify the method of calculating the voltage drop across it. Experimentally the voltage drop is measured by determining the alteration in a magnetic flux coupled with the conductor $u_L = L_{p0}c^{-2}\mathrm{d}i/\mathrm{d}t$ (L_{p0} is the external inductance of the conductor) and also the Ohmic voltage drop that represents an integral of the z component of the electric field on the conductor surface $E_z(r_0, z)$: $u_R = \int_0^l E_z(r_0, z)\,\mathrm{d}z$. Then, with consideration of the boundary conditions, the voltage drop across the conductor is

$$u(t) = u_L(t) + u_R(t) = L_{p0}I_0(c^2t_0)^{-1}\dot{I} + R_{p0}I_0(I - (\pi r_1)^{-1}J_0^2(g_1)Z). \tag{48}$$

In equation (48) $t_0 = r_0^2b(4g_1^2\nu_m)^{-1}$ is the base time; $R_{p0} = l(\pi r_0^2\sigma)^{-1}$ is the initial conductor resistance. It is evident that to close system (41)–(43), it is necessary to specify an equation (or equations) for determining the current I, the form of which depends on a concrete topology of an external circuit that serves as a thermostat with respect to the dynamic system (41)–(43).

4. Discussion

If we consider that $\pi g_1^{-1} \cong 1$, then equations (41) and (42) agree fully with the first two equations in the Lorenz model [10], and (43) has the sign '−' before XY instead of sign '+' in the corresponding Lorenz equation. Thus our model describes NPT, as will be shown below.

Let us find a spatial scale of perturbation in the z direction, which allows us to discuss the suitability of the assumption made when deriving (41)–(43) that the minimal number of perturbation modes is three. We make use of the dependence of the critical Rayleigh number on the parameter b (see also figure 1): $R_c = 64g_1^2\pi^2(b^2(4-b))^{-1}$. We can see that $R_c \to \infty$ as $b \to 0$ or $b \to 4$. The case $b = 0$ corresponds to a shortwave limit, i.e. $k = \infty$ ($\lambda = 0$). The case $b = 4$ corresponds to a longwave limit when $k = 0$ and $\lambda = \infty$. Hence, in the latter case, the perturbation spreads over the whole volume occupied by the matter.

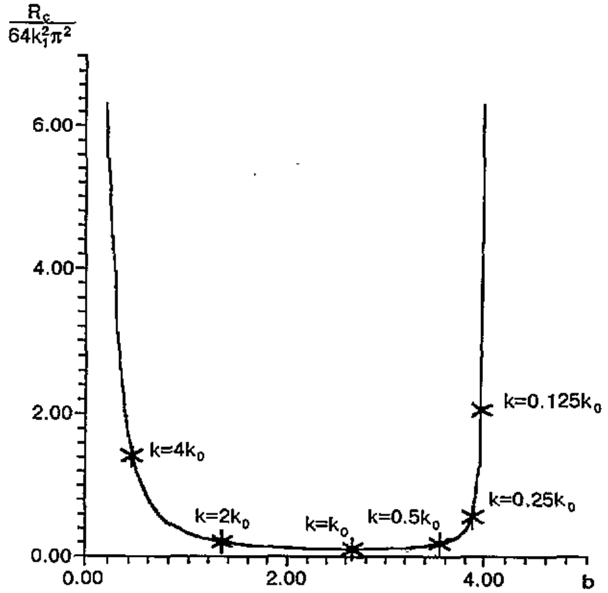

Figure 1. The magnetic Rayleigh number as a function of the parameter b.

In our case, a characteristic scale of the problem is the conductor radius r_0. Therefore it should be expected that the most rapidly developed perturbations are those with $\lambda \cong r_0$.

Actually, R_c has a minimum corresponding to the condition $\mathrm{d}R_c/\mathrm{d}b = 0$: $b = 8/3$, $R_{c,\min} = 978.08144$. This value of the Rayleigh critical number corresponds to the perturbation with $\lambda = r_0k^{-1} = r_0\pi g_1{}^{-1}b^{1/2}(4-b)^{-1/2} = 1.15931r_0$. Respectively, $r_1 = 1.0224\times10^{-3}R_c = 6.5088\times10^{-4}I_0{}^2(c^2\rho_0\nu_m\nu)^{-1}$. A subsequent perturbation evolution is possible in the direction of splitting the spatial scale. In addition, as shown in figure 1, the process of splitting the conductor into strata having the size $l = 2\lambda$ (the Rayleigh critical number for a perturbation with the wavenumber $0.5k$ is 1.65×10^3 and, for a perturbation with $l = 0.5\lambda$, it is 1.957×10^3) is more advantageous at first. This analysis is supported by a direct observation of the conductor stratification in EEC experiments [3, 4] (figure 2 (taken from [11]) shows that in the final explosion stage the conductor splits into strata with size $l \cong 2\lambda$). In experiments with copper conductors 0.58 mm in diameter [4], the mean distance between the strata was 0.78 mm. According to our model, $l \cong 2\lambda = 0.672$ mm, which is in rather good agreement with experiment (we did not use any experimental data while calculating λ). The consideration of a larger number of perturbation modes does not show a significant change in the results of the above analysis, as modes with the wavenumber not equal to k have a longer time of perturbance than the dominant mode. The fact that $R_{c,\min}$ corresponds to $\lambda \cong r_0$ shows evidence that the model proposed is a model of a non-equilibrium phase transition where the amplitudes $X(t)$, $Y(t)$, $Z(t)$ and $I(t)$ have a physical meaning of the order parameters, the interaction of which is described by (41)–(43) followed by one or several equations for finding the current $I(t)$ (depending on a concrete topology of an external electric circuit).

Let us analyse the set of equations (41)–(43) at $I = 1 =$ constant. We have

$$\dot{\mathbf{W}} = \mathbf{F}(\mathbf{W}) \qquad \mathbf{W} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{49}$$

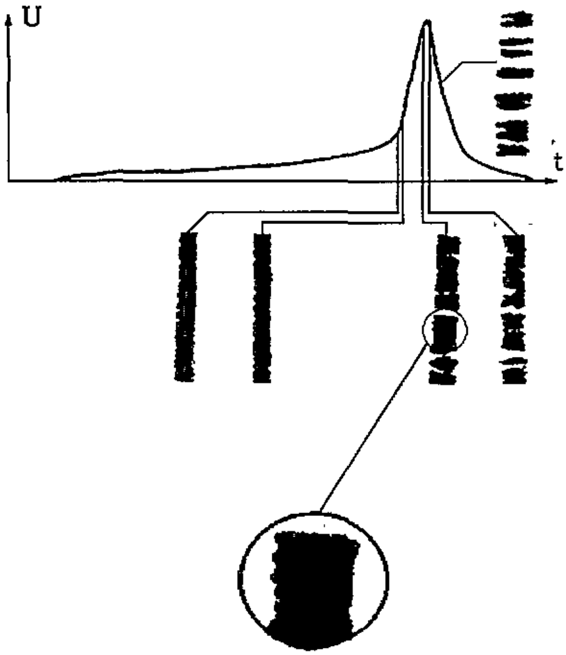

Figure 2. X-ray patterns of an exploding conductor at different times (from the experiment in [11]).

The fixed points of the set (49) are determined by the condition $\mathbf{F}(\mathbf{W}) = \mathbf{0}$: $\mathbf{W}_1 = \mathbf{0}$, $\mathbf{W}_{2,3} = \{\pm 1.221[b(1-0.672r_1)]^{1/2}; \pm 1.221[b(1-0.672r_1)]^{1/2}; -1.221(1-0.672r_1)\}$. One can see that at $r_1 > r_* = 1.488$, the set (49) has no fixed points. The analysis of the solution stability shows that at $r_1 < r_*$ the fixed points are stable. The fixed point corresponding to $r_1 = r_*$ is a stability boundary (figure 3(*a*) shows a bifurcation curve for the set (49)).

The coordinates of the fixed points of the Lorenz model are $\mathbf{W}_1 = \mathbf{0}$, $\mathbf{W}_{2,3} = \{\pm[b(r_1 - 1)]^{1/2}; \pm[b(r_1 - 1)]^{1/2}; r_1 - 1\}$. The critical point of the Lorenz model ($r_1 = RR_c{}^{-1} = 1$) is also a stability boundary (figure 3(*b*) shows a bifurcation curve of the Lorenz model). One can see that at $R > R_c$ there are stable steady-state solutions in the Lorenz model; a transition from the point $\mathbf{W} = \mathbf{0}$ to any arbitrary point of the phase space being

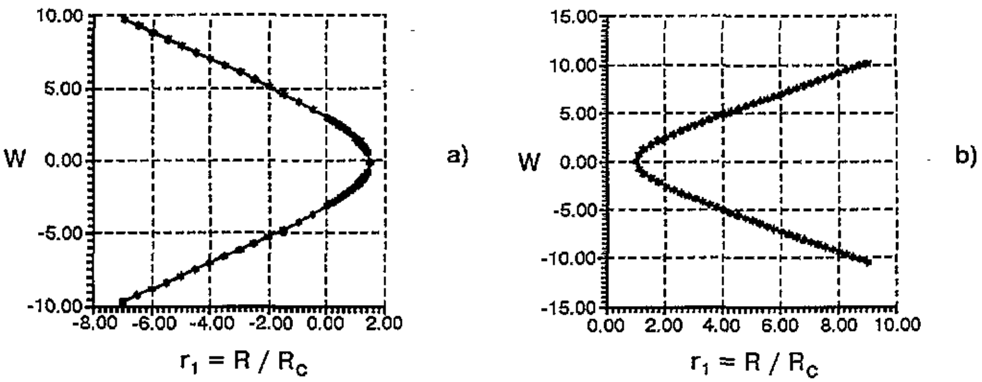

Figure 3. The bifurcation curves for our model (*a*), and for the Lorenz one (*b*).

performed via the adiabatic sequence of stable states. A bifurcation, called in the literature a supercritical bifurcation (the Hopf bifurcation), occurs at a critical point [12].

In our case, the bifurcation is a subcritical one, as at $R > R_c$ there are no steady-state solutions near the critical point. In addition, any trajectory of the system in phase space at $R > R_c$ shifts the system from the point $\mathbf{W} = \mathbf{0}$ to infinity during a finite time. In literature this transition is called explosive [12].

The asymptotic behaviour of the solution of set (49) is of a singular character

$$X \sim a_1(t_* - t)^{-1} \qquad Y \sim a_2(t_* - t)^{-2} \qquad Z \sim a_3(t_* - t)^{-2} \tag{50}$$

where $a_1 = \pm 2.442$, $a_2 = \pm 2.442 s^{-1}$, $a_3 = -2.442 s^{-1}$. The Lorenz model has a similar asymptotic behaviour though with different factors to (50); $\{a_i, i = 1, 2, 3\}$: $a_1 = 2\mathrm{i}$, $a_2 = -2\mathrm{i}s$, $a_3 = -2s$ (an imaginary unit in factors a_1 and a_2 indicates a fluctuating character in a steady-state solution of the Lorenz model) [13].

It is worthwhile coming back to the interpretation of the Rayleigh magnetic number and the control parameter $r_1 = R R_c^{-1}$: $R = v_A{}^2 r_0{}^2 (\nu_m \nu)^{-1}$. Sausage-type MHD instabilities are known to be an alternative mechanism when a conductor is destroyed by current, the increment of which is $\gamma_A = t_A{}^{-1} = 2^{1/2} v_A r_0{}^{-1}$ (t_A is the time for the development of a MHD instability) [14]. Then the Rayleigh magnetic number can be represented as a square ratio of the times t_d and t_A: $R = (t_d t_A{}^{-1})^2$ ($t_d = r_0{}^2 (2\nu_m \nu)^{-1}$ is the effective diffusion time). For $I = 1$, the equality $R = 1.488 R_c$ corresponds to the critical point on the bifurcation curve, whereas the inequality $t_d t_A{}^{-1} \geqslant (1.488 R_c)^{1/2} = 38.1495$ corresponds to the supercritical regime in our model.

On the other hand sausage-type MHD instabilities are known to develop when the magnetic pressure $P_m = H^2 (8\pi)^{-1}$ surpasses the pressure in the medium. Let us estimate critical values of the magnetic intensity at a constant current. The first value H_{cr1} is determined to correspond to the Rayleigh critical number $R_c = H_{cr1}{}^2 (2\pi \rho_0 \nu_m \nu)^{-1} = 978.081\,44$: $H_{cr1} = 78.393 r_0{}^{-1} (\nu_m \nu)^{1/2}$. Setting $\sigma \cong 10^{17} c^{-1}$, we obtain $H_{cr1} = 2.1 \times 10^3 r_0{}^{-1} \eta^{1/2}$ Oe. The second value of the critical field intensity (H_{cr2}) is determined from the pinch condition $H_{cr2}{}^2 (8\pi P)^{-1} \geqslant 1$. Setting $P \cong \rho_0 c_s{}^2$, $\rho_0 \cong 10$ g cm^{-3} and $c_s \cong 10^5$ cm s^{-1}, we obtain $H_{cr2} = 1.585 \times 10^6$ Oe. Thus, at the conductor radius $r_0 = 10^{-2}$ cm widely used in experiment and the overrated shear viscosity factor $\eta \cong 1$ P, $H_{cr1} < H_{cr2}$.

The control parameter $r_1 = R R_c{}^{-1} = (H H_{cr1}{}^{-1})^2 > 1.488$. Hence, in our model a non-equilibrium phase transition develops under the condition $H \geqslant 2.561\,65 \times 10^3 r_0{}^{-1} \eta^{1/2}$ Oe. At $r_0 \cong 10^{-2}$ cm and $\eta \cong 1$ P (six!) the following inequality is valid

$$2.561\,65 \times 10^5 \text{ Oe} < H < 1.585 \times 10^6 \text{ Oe}.$$

Let us find the conductor radius (r_*), when $H_{cr1} = H_{cr2} = H$. We obtain $r_* = 1.616 \times 10^{-3} \eta^{1/2} < 1.616 \times 10^{-3}$ cm. Since the actual liquid metal viscosity is about 10^{-2} P, our calculations are overestimated by at least an order of magnitude.

Consequently, the instability described by our model can be referred to as a class of magnetohydrodynamic instabilities, with its excitation threshold being sufficiently lower than that of ordinary sausage-type instabilities and therefore, to investigate the former, it is not necessary, unlike in [14], to disturb the conductor surface. Moreover, vortex structures developed in the conductor during their further evolution are acting on the conductor surface to be drawn in, which results in the conductor splitting (to describe the initial phase of splitting, we suggested a three-mode model [15], similar to (41)–(43), according to which the X, Y and Z amplitudes have a time singularity: $X \sim Y \sim Z \sim (t_* - t)^{-1/2}$ and

$I \sim (t_* - t)^{1/2}$; it should be noted that the dynamics of the process in this stage is quantitatively similar to the destruction of superconductivity by a critical current). As an additional argument in favour of the transition to splitting, the following can be stated: at a constant current the power $P = UI$ has the asymptote $P \sim (t_* - t)^{-2}$, in splitting it is $P \sim (t_* - t)^0 =$ constant. Thus, it is more energetically beneficial to split a spatial scale than to increase the X, Y and Z amplitudes preserving the initial value of the scale.

5. Conclusions

The principal results of the present paper can be summarized as follows:

- The initial model of the magnetic hydrodynamics of a conducting fluid has been transformed to the form where the possibility of large-scale structure formations is presented in an explicit form; both the 'heavy' and the 'light' fluid (conducting electrons) consists of two fluids: laminar and vortex. The increase in an effective conductor resistance even in the case of a constant local conductance is a consequence of the vortex structure formation in the electron subsystem (electric current).
- In the framework of the present model, a model of a non-equilibrium phase transition has been derived where the motion of a laminar component of a heavy fluid was not taken into account and kinetic coefficients were considered as constant, i.e. independent of density and temperature. This model can be referred to as a class of the Lorenz model [10], the spatial scale of a vortex structure being in good agreement with experiment [4].
- The instability discussed in the paper has been shown to belong to a class of magnetohydrodynamic instabilities. It is characterized by a threshold current which is at least a factor of 10^2 lower than that for an ordinary sausage-type instability. There is also no need for a disturbance of the conductor surface to study this instability. Moreover, in the case of a free conductor surface, the spatial scale splitting has been shown to be inevitable as a consequence of the initial vortex structure formation and evolution.

It should be noted that from the moment when the spatial scale starts splitting, the model presented by (41)–(43) loses its applicability. Therefore, our next problem is to develop models for a sequence of doubling the spatial scale, mentioned in section 3: $k_0 \rightarrow k_1 = 0.5k_0 \rightarrow k_2 = 2k_0 \rightarrow k_3 = 2k_2 \rightarrow k_4 = 2k_3$, etc. In the stage $k_0 \rightarrow k_1$, as shown in [15], we can also confine ourselves to three modes of perturbation.

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